SYNTHESIS OF RECURRENT ALGORITHMS OF THE DATA INTERPRETATION

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Introduction

In many aerodynamic experiments, the mathematical model of measurements is the following equation

$$\tilde{f}_n = \theta \cdot k_n + \eta_n, \quad n = 1,2,..., \tag{1}$$

where θ is the unknown vector of dimension M, k_n is the "scattering" function of the measuring system (string-vector of dimension M), η_n is the noise of measurements with zero average value and dispersion σ_n^2. The problem of interpretation of such measurements consists in construction of an estimation $\theta^{(n)}$ for the vector θ on the basis of measurements $\tilde{f}_i, i = 1, 2, ..., n$. This problem is solved using recurrent algorithms, in which an update of the estimation $\theta^{(n-1)}$ (constructed on the previous step) is implemented at each new measurement $\tilde{f}_n$. The recurrent algorithm is described by equations

$$\theta^{(n)} = \theta^{(n-1)} + \frac{P^{(n)}}{\sigma_n^2} k_n^T (\tilde{f}_n - k_n \theta^{(n-1)}); \tag{2}$$

$$P^{(n)} = P^{(n-1)} - \frac{P^{(n-1)} k_n^T k_n P^{(n-1)}}{\sigma_n^2 + k_n P^{(n-1)} k_n^T}, \quad n = 1, 2, ..., \tag{3}$$

with a start point $\theta^{(0)}, P^{(0)}$. A zero vector is often assumed as a vector $\theta^{(0)}$. A matrix $P^{(0)}$ of dimension $M \times M$ is set by the expression $\alpha^{-1} \cdot I$, where α is the regularization parameter and I is the unit matrix. It is evident that the error of the estimate $\theta^{(n)}$ depends on the step number n. However, the choice of the recurrent algorithm stop moment is considered in the literature rather insufficiently. Therefore a question arises "How to choose the number n for which the norm of the solution error was minimum or recurrent algorithm meets certain requirements". To answer this question, the accuracy characteristics of the recurrent algorithm (2), (3) used to solve the problem of algorithm synthesis are introduced. It should be noted, that the solution of this problem allows one to find the least number of the measurements, which is necessary to obtain the estimate $\theta^{(n)}$ with the required accuracy.

Accuracy characteristics of the recurrent algorithm

At each step, the vector of the estimate error is defined by the expression

$$\varepsilon^{(n)} = \theta^{(n)} - \theta,$$

Report Documentation Page

Report Date	Report Type	Dates Covered (from... to)
23 Aug 2002	N/A	-

Title and Subtitle Synthesis of Recurrent Algorithms of the Data Interpretation	**Contract Number**
	Grant Number
	Program Element Number
Author(s)	**Project Number**
	Task Number
	Work Unit Number
Performing Organization Name(s) and Address(es) Institute of Theoretical and Applied Mechanics Institutskaya 4/1 Novosibirsk 530090 Russia	**Performing Organization Report Number**
Sponsoring/Monitoring Agency Name(s) and Address(es) EOARD PSC 802 Box 14 FPO 09499-0014	**Sponsor/Monitor's Acronym(s)**
	Sponsor/Monitor's Report Number(s)

Distribution/Availability Statement
Approved for public release, distribution unlimited

Supplementary Notes
See also ADM001433, Conference held International Conference on Methods of Aerophysical Research (11th) Held in Novosibirsk, Russia on 1-7 Jul 2002

Abstract

Subject Terms

Report Classification unclassified	**Classification of this page** unclassified
Classification of Abstract unclassified	**Limitation of Abstract** UU

Number of Pages
6

where θ is the "exact" vector of unknown parameters. The vector $\varepsilon^{(n)}$ may be written as a sum of two components $\varepsilon^{(n)} = \xi^{(n)} + b^{(n)}$, where $\xi^{(n)} = \theta^{(n)} - \overline{\theta}^{(n)}$ is the vector of the random error caused by "transferring" to the solution $\theta^{(n)}$ the random error of initial data; $b^{(n)} = \overline{\theta}^{(n)} - \theta$ is the vector of the systematic error. The vector $\overline{\theta}^{(n)}$ is the solution of system (2), (3) for exact values $f_n = k_n \cdot \theta$. Let us consider the calculation of these vectors.

We will enter a rectangular matrix $B(n)$ of dimension $M \times M$ involved in the expression

$$b^{(n)} = B(n) \cdot (\theta^{(0)} - \theta) = B(n) b^{(0)},$$

where $b^{(0)} = \theta^{(0)} - \theta$ is the vector of "initial" bias. We will call $B(n)$ *a bias matrix*.

Statement 1. The matrix $B(n)$ for the recurrent algorithm (2), (3) is defined by the expression

$$B(n) = \prod_{j=0}^{n-1} \left(I - \frac{P^{(j)} k_{j+1}^T k_{j+1}}{k_{j+1} P^{(j)} k_{j+1}^T + \sigma_{j+1}^2} \right)$$

and allows for recurrent calculation:

$$B(n) = \left(I - \frac{P^{(n-1)} k_n^T k_n}{k_n P^{(n-1)} k_n^T + \sigma_n^2} \right) \cdot B(n-1), \; n = 1, 2, \ldots$$

where $B(0) = I$.

We will take the *correlation matrix* $V_\xi(n) = M\left[\xi^{(n)} \xi^{(n)^T} \right]$ as a characteristic for the vector of a random error $\xi^{(n)}$, where $M[\;\;]$ is the operator of mathematical expectation with respect to the noise distribution.

Statement 2. The correlation matrix $V_\xi(n)$ allows the recurrent calculation:

$$V_\xi(n) = (I - G(n) \cdot k_n) \cdot V_\xi(n-1) \cdot (I - G(n) \cdot k_n)^T + \sigma_n^2 \cdot G(n) \cdot G^T(n), \tag{4}$$

$$V_\xi(0) = 0, \; n = 1, \; 2, \ldots,$$

where

$$G(n) = \frac{P^{(n-1)} k_n^T}{k_n P^{(n-1)} k_n^T + \sigma_n^2}.$$

The introduced bias matrix $B(n)$ and correlation matrix $V_\xi(n)$ characterize the systematic and random errors sufficiently. However, in order to simplify the synthesis, we should introduce some additional characteristics of these matrices.

It can be easily shown that *the root-mean-square error* (rms) of the estimate $\theta^{(n)}$ is defined by the formula

$$\Delta^2(n) = M\left[\left\| \theta^{(n)} - \theta \right\|^2 \right] = b^{(n)^T} b^{(n)} + Sp\left[V_\xi(n) \right], \tag{5}$$

where $Sp[\;\;]$ is the matrix trace. Let us consider each addend in (5) separately.

In practice it is impossible to calculate the vector $b^{(n)}$, because we do not know the "initial" bias, that is, the vector $b^{(0)}$. Therefore we shall define the scalar characteristic $U_b(n)$ [1], defined as

$$U_b(n) = \frac{\left\| b1^{(n)} \right\|^2}{\left\| 1_{M\times 1} \right\|^2},$$

where $1_{M\times 1}$ is the unit vector, $b1^{(n)}$ is the bias vector stipulated by a unit vector of initial bias, that is, $b^{(0)} = 1_{M\times 1}$. We can show that

$$U_b(n) = \frac{1}{M}\sum_{i=1}^{M}\left(\sum_{j=1}^{M} \{B(n)\}_{i,j} \right)^2 .$$

The following limits are valid:

$$\lim_{n\to 0} U_b(n) = 1, \quad \lim_{n\to\infty} U_b(n) = 0 .$$

Let us consider the second addend in (5). We will determine the scalar characteristic $U_\xi(n)$ [1], defined as

$$U_\xi(n) = \frac{Sp\left[V_\xi(n) \right]}{\sigma_n^2},$$

that can be interpreted as a coefficient of transference of noise variance to the quantity

$$M\left[\left\| \xi_\alpha^{(n)} \right\|^2 \right] = Sp\left[V_\xi(n) \right].$$

As it follows from (4), the algorithm for calculating $U_\xi(n)$ can be defined:

$$U_\xi(n) = \frac{1}{\sigma_n^2} Sp[(I - G_\alpha(n) \cdot k_n) \cdot V_\xi(n-1) \cdot (I - G_\alpha(n) \cdot k_n)^T] + Sp\left[G_\alpha(n) \cdot G_\alpha^T(n) \right].$$

Using the characteristics $U_b(n)$, $U_\xi(n)$, we can approximate the MSE of estimation $\theta^{(n)}$ by the expression

$$\Delta^2(n) \cong U_b(n) \cdot \left\| b^{(0)} \right\|^2 + U_\xi(n) \cdot \sigma_n^2 . \tag{6}$$

The values $U_b(n)$ and $U_\xi(n)$ determine systematic and random errors too full and they may be called *the accuracy characteristics of* the recurrent algorithm (2), (3).

Let us consider the results of one computing experiment**.** The vector θ, having 30 projections, is measured using (1). The number of measurements N is equal to 80. The condition number of a the measurement matrix is equal $2\cdot 10^7$. The relative level of measurement noise is equal 0.05. Figures 1 and 2, respectively, demonstrate U_b, U_ξ as functions of the iteration number n. They show the plots for two regularization parameters: $\alpha = 10^{-1}$ (plot 1) and $\alpha = 10^{-7}$ (plot 2). The behaviour of $U_b(n)$ and $U_\xi(n)$ demonstrate a known *inconsistency between the stability and the resolution of the algorithm.* With increase

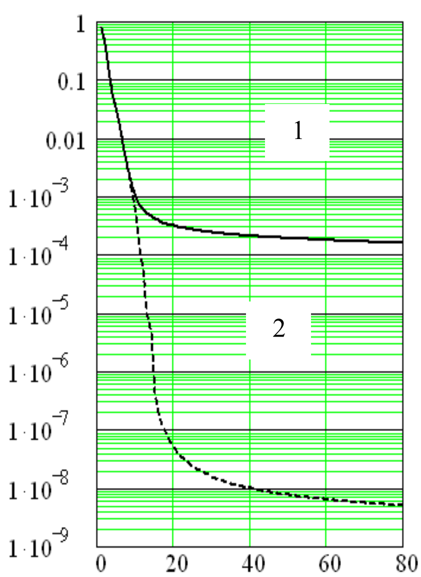

Fig. 1

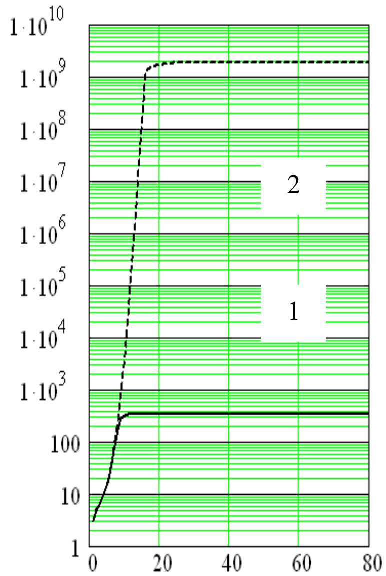

Fig. 2

of the iteration number *n the resolution is improved*, but *the stability of the estimation* $\theta^{(n)}$ *to measurement noise is worsened.*

It is evident, that calculating these characteristics we can perform an accuracy analysis of the recurrent algorithm (2), (3). For this purpose in formula (6), instead of an unknown quantity $\left\| b^{(0)} \right\|^2$, its estimation $b_{\max}^2$ is substituted. To illustrate this possibility, the plots of the functions

$$\Delta_1^2(n) = U_b(n) \cdot b_{\max}^2 + U_\xi(n) \cdot \sigma_n^2 ; \tag{7}$$

$$\Delta_2^2(n) = \left\| \theta^{(n)} - \theta \right\|^2 ,$$

calculated for the previously described measurements (see Fig.1 and Fig.2) are presented in the Fig. 3. The value of $b_{\max}^2$ was 30 % more than the value $\left\| b^{(0)} \right\|^2$. From Fig. 3 it is visible that the minimum point of the function $\Delta_2^2(n)$ (random error of the estimation $\theta^{(n)}$) is in a neighbourhood of the minimum point of the function $\Delta_1^2(n)$ (expectation of the random error of estimation), and therefore, $\Delta_1^2(n)$ can be used to analyze the recurrent algorithm (2), (3).

An important property of $\Delta_1^2(n)$, $U_b(n), U_\xi(n)$ is the possibility of their *"a priori" calculation*, since the measurements $\tilde{f}_i$ are not involved in calculation of the accuracy characteristics. This allows us to determine the iteration number proceeding from the required

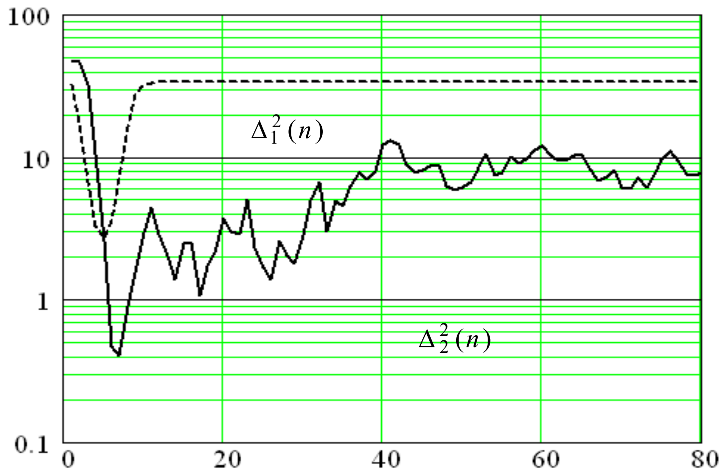

Fig. 3

values of these characteristics, i.e. perform *synthesis of the recurrent algorithm*. For this purpose, we will formulate the following *variational problems*:

Problem A

$$\min_{n>0} \Delta_1^2(n).$$

Problem B

$$\min_{n>0} U_b(n) \quad \textit{under constraint} \quad U_\xi(n) \le U_\xi^{\max}.$$

Problem C

$$\min_{n>0} U_\xi(n) \quad \textit{under constraint} \qquad U_b \le U_b^{\max}.$$

Let us note the following:

– The solution of problem A (denote it as n_A) is an estimate of the iteration number at which the rms error of the estimate $\theta^{(n)}$ is minimum, but as *a priori information* we must specify the values of $\left\|b^{(0)}\right\|^2$ and σ_n^2. As a rule, these values are unknown. Therefore, we shall construct a recurrent algorithm, which will be optimum both on classes of vectors θ and measurement noise satisfying the following limitations:

$$\left\| b^{(0)} \right\|^2 \le b_{\max}^2; \quad \sigma_n^2 \le \sigma_{\max}^2. \tag{8}$$

Substituting $b_{\max}^2, \sigma_{\max}^2$ into (7) and solving the problem A we find the n_A, that is, the number of steps minimizing the rms error of the estimate $\theta^{(n)}$ on classes (8).

– The solution of the problem B (denote it by n_B) minimizes the systematic error with the required stability to the measurement noise .

– The solution of the problem C (denote it by n_C) minimizes the random error with the required algorithm resolution.

– Taking into account the monotone behavior of the characteristics $U_b(n), U_\xi(n)$, the solution of the variational problem B is determined as the root of the nonlinear equation

$$U_\xi(n_B) \cong U_\xi^{\max},$$

and the solution of the variational problem C - as the root of the nonlinear equation

$$U_b(n_C) \cong U_b^{\max}.$$

The sign $\cong$ shows that the solution is sought among the integer values of the arguments $U_b(n)$ and $U_\xi(n)$.

Conclusion

The proposed accuracy characteristics of the recurrent algorithm (2),(3) allow us to perform not only its analysis but also the synthesis, i.e., to choose the iteration number of the recurrent algorithm stopping, at which the desired accuracy characteristics are ensured.

REFERENCES

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